Eigenmath Manual

August 1, 2007

Math	*Eigenmath*	*Alternate form and/or comment*
$-a$	`-a`	
$a+b$	`a+b`	
$a-b$	`a-b`	
ab	`a*b`	`a b` *with a space in between*
$\frac{a}{b}$	`a/b`	
$\frac{a}{bc}$	`a/b/c`	
a^2	`a^2`	
$\sqrt{a}$	`a^(1/2)`	`sqrt(a)`
$\frac{1}{\sqrt{a}}$	`a^(-1/2)`	`1/sqrt(a)`
$a(b+c)$	`a*(b+c)`	`a (b+c)` *with a space in between*
$f(a)$	`f(a)`	
$\begin{pmatrix} a \\ b \\ c \end{pmatrix}$	`(a,b,c)`	
$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$	`((a,b),(c,d))`	
T^{12}	`T[1,2]`	*tensor component access*
$2\,\mathrm{km}$	`2*"km"`	*units of measure should be quoted*

Let us begin by considering the following passage from Vladimir Nabokov's autobiography *Speak, Memory.*

> A foolish tutor had explained logarithms to me much too early, and I had read (in a British publication, the *Boy's Own Paper*, I believe) about a certain Hindu calculator who in exactly two seconds could find the seventeenth root of, say, 3529471145 7602751323018973420558661713 92 (I am not sure I have got this right; anyway the root was 212).

We can check Nabokov's arithmetic by typing the following into Eigenmath.

```
212^17
```

After pressing the return key, Eigenmath displays the following result.

$$352947114576027513230189734205586617139 2$$

So Nabokov did get it right after all. We can enter *float* or click on the float button to scale the number down to size.

```
float
```

$$3.52947 \times 10^{39}$$

A geometric series converges according to the formula

$$\sum_{k=0}^{\infty} a^k = \frac{1}{1-a}, \qquad |a| < 1$$

If we use $a = -1/2$ and for practical purposes only count up to nine instead of infinity, we should have

$$\sum_{k=0}^{9} \left(-\frac{1}{2}\right)^k \approx \frac{2}{3}$$

In the following examples, various Eigenmath features will be demonstrated by computing this formula in different ways.

To begin, here is the calculation in one line of code.

```
sum(k,0,9,(-0.5)^k)
```

$$0.666016$$

The following example uses an intermediate variable.

```
f=sum(k,0,9,a^k)
f
```

$$f = 1 + a + a^2 + a^3 + a^4 + a^5 + a^6 + a^7 + a^8 + a^9$$

```
eval(f,a,-1/2)
```

$$\frac{341}{512}$$

```
float(last)
```

$$0.666016$$

As seen on the first line, no result is printed when a symbol is defined. When you do in fact want to see the value of a symbol, just enter it as shown on the second line.

When a result is displayed, it is also stored in the symbol *last*.

The following example shows how to define a function.

```
f(a)=sum(k,0,9,a^k)
f
```

$$f = \text{sum}(k, 0, 9, a^k)$$

```
f(-1/2)
```

$$\frac{341}{512}$$

```
f(-0.5)
```

$$0.666016$$

Eigenmath handles function definitions in a special way. Unlike a normal symbol, a function definition is not evaluated immediately. The following example demonstrates the difference.

```
f=sum(k,0,9,a^k)
f
```

$$f = 1 + a + a^2 + a^3 + a^4 + a^5 + a^6 + a^7 + a^8 + a^9$$

```
f(a)=sum(k,0,9,a^k)
f
```

$$f = \text{sum}(k, 0, 9, a^k)$$

Why are they handled differently? Well, suppose we want to define f as a function of two variables, a and n, like this.

```
f(a,n)=sum(k,0,n,a^k)
```

In this case it is not possible to evaluate the sum right away. The value of n will not be known until f is actually used somewhere. Consequently, Eigenmath does not normally evaluate a function when it is defined. Now, having said that, there are two work arounds that provide exceptions to the rule. They are *eval* and *quote*.

```
f=quote(sum(k,0,9,a^k))
f
```

$$f = \text{sum}(k, 0, 9, a^k)$$

```
f(a)=eval(sum(k,0,9,a^k))
f
```

$$f = 1 + a + a^2 + a^3 + a^4 + a^5 + a^6 + a^7 + a^8 + a^9$$

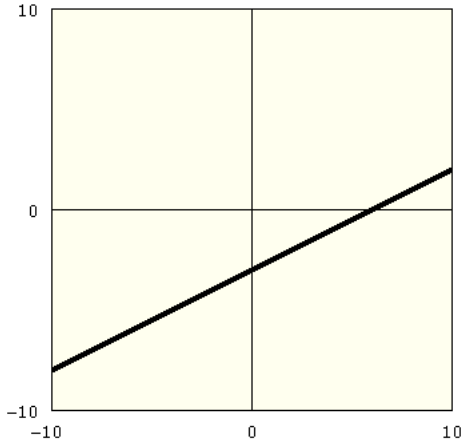
Quoted strings can be used to express units of measurement in a calculation. For example, the space shuttle accelerates from zero to 17,000 miles per hour in 8 minutes. The average acceleration of the space shuttle is

```
v=17000*"mile"/"hr"
t=8*"min"/(60*"min"/"hr")
v/t
```

$$\frac{127500\ \text{mile}}{(\text{hr})^2}$$

Here is a simple example that draws the graph of $y = mx + b$.

```
y=m*x+b
m=1/2
b=-3
draw(y)
```

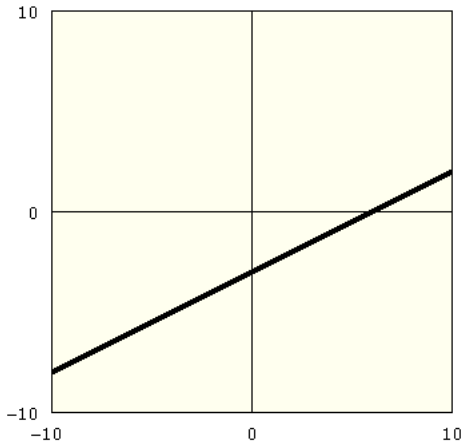


Now suppose that we want to draw the graph with a different m. We could type in everything all over again, but it would be easier in the long run to write a script. Then we can go back and quickly change m and b as many times as we want.

To prepare a script, click on the Edit Script button. Then enter the script commands, one per line.

```
y=m*x+b
m=1/2
b=-3
draw(y)
```

Next, click on the Run Script button to see the graph.



Eigenmath runs a script by stepping through it line by line. Each line is evaluated just like a regular command. This continues until the end of the script is reached. After the script runs, you can click Edit Script and go back and change something.

By the way, Eigenmath automatically does a clear before running a script.

Sometimes it is desirable to have a script print a few comments when it runs. This can be accomplished by placing the desired text in quotes on a single line. For example, the script

```
"Here is the value of pi."
float(pi)
```

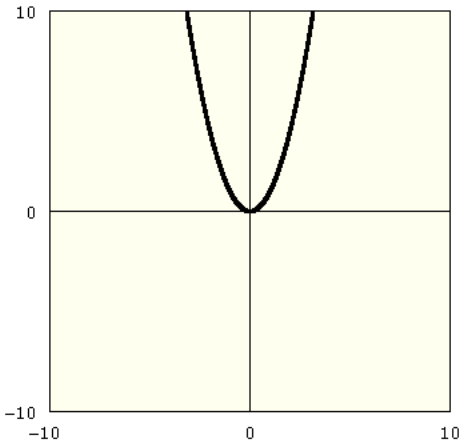
displays the following when run.

```
Here is the value of pi.
```

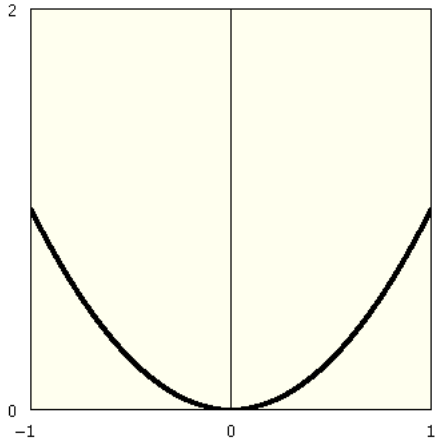
3.14159

draw(f, x) draws a graph of the function f of x. The second argument can be omitted when the dependent variable is literally x or t. The vectors *xrange* and *yrange* control the scale of the graph.

```
draw(x^2)
```




```
xrange=(-1,1)
yrange=(0,2)
draw(x^2)
```

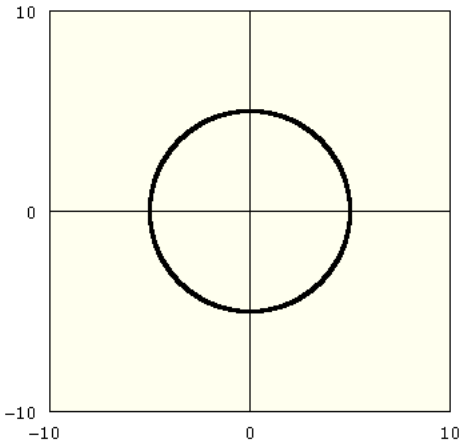



```
clear
```

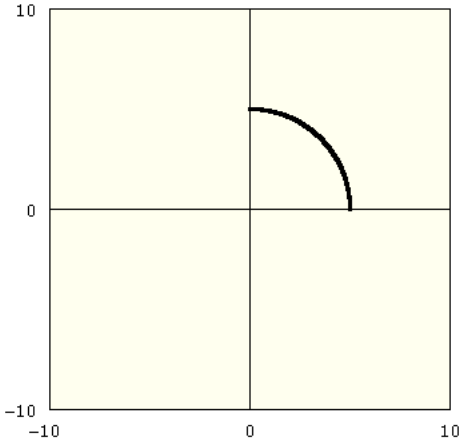
The clear command (or a click of the Clear button) resets *xrange* and *yrange*. This needs to be done so that the next graph appears as shown.

Parametric drawing occurs when a function returns a vector. The vector *trange* controls the parameter range. The default range is $(-\pi, \pi)$.

```
f=(cos(t),sin(t))
draw(5*f)
```

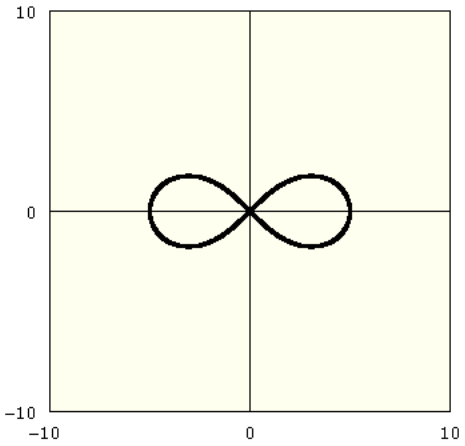



```
trange=(0,pi/2)
draw(5*f)
```



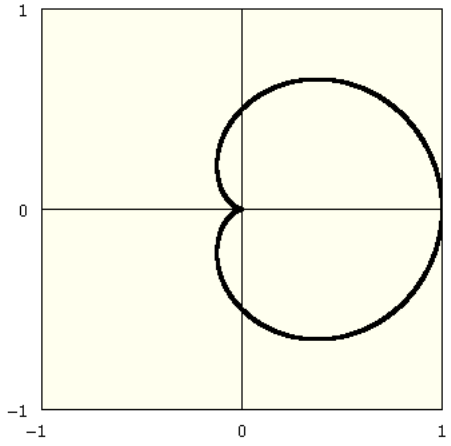
Here are a couple of interesting curves and the code for drawing them. First is a lemniscate.

```
clear
X=cos(t)/(1+sin(t)^2)
Y=sin(t)*cos(t)/(1+sin(t)^2)
draw(5*(X,Y))
```



Next is a cardioid.

```
r=(1+cos(t))/2
u=(cos(t),sin(t))
xrange=(-1,1)
yrange=(-1,1)
draw(r*u)
```



dot is used to multiply vectors and matrices. The following example shows how to use *dot* and *inv* to solve for $\mathbf{X}$ in $\mathbf{AX} = \mathbf{B}$.

```
A=((3.8,7.2),(1.3,-0.9))
B=(16.5,-22.1)
X=dot(inv(A),B)
X
```

$$\begin{pmatrix} -11.2887 \\ 8.24961 \end{pmatrix}$$

One might wonder why the *dot* function is necessary. Why not simply use $X = inv(A) * B$ like scalar multiplication? The reason is that the software normally reorders factors internally to optimize processing. For example, $inv(A) * B$ in symbolic form is changed to $B * inv(A)$ internally. Since the dot product is not commutative, this reordering would give the wrong result. Using a function to do the multiply avoids the problem because function arguments are not reordered.

It should be noted that *dot* can have more than two arguments. For example, $dot(A, B, C)$ can be used for the dot product of three tensors.

The following example demonstrates the relation $\mathbf{A}^{-1} = \operatorname{adj}\mathbf{A}/\det\mathbf{A}$.

```
A=((a,b),(c,d))
```

```
inv(A)
```

$$\begin{pmatrix} \dfrac{d}{ad-bc} & -\dfrac{b}{ad-bc} \\ -\dfrac{c}{ad-bc} & \dfrac{a}{ad-bc} \end{pmatrix}$$

```
adj(A)
```

$$\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

```
det(A)
```

$$ad - bc$$

```
inv(A)-adj(A)/det(A)
```

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Sometimes a calculation will be simpler if it can be reorganized to use *adj* instead of *inv*. The main idea is to try to prevent the determinant from appearing as a divisor. For example, suppose for matrices $\mathbf{A}$ and $\mathbf{B}$ you want to check that

$$\mathbf{A} - \mathbf{B}^{-1} = 0$$

Depending on the complexity of $\det\mathbf{B}$, the software may not be able to find a simplification that yields zero. Should that occur, the following alternative can be tried.

$$(\det\mathbf{B}) \cdot \mathbf{A} - \operatorname{adj}\mathbf{B} = 0$$

The adjunct of a matrix is related to the cofactors as follows.

```
A=((a,b),(c,d))
C=((0,0),(0,0))
C[1,1]=cofactor(A,1,1)
C[1,2]=cofactor(A,1,2)
C[2,1]=cofactor(A,2,1)
C[2,2]=cofactor(A,2,2)
C
```

$$C = \begin{pmatrix} d & -c \\ -b & a \end{pmatrix}$$

```
adj(A)-transpose(C)
```

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

When Eigenmath starts up, it defines the symbol i as $i = \sqrt{-1}$. Other than that, there is nothing special about i. It is just a regular symbol that can be redefined and used for some other purpose if need be.

Complex quantities can be entered in rectangular or polar form.

```
a+i*b
```

$$a + ib$$

```
exp(i*pi/3)
```

$$\exp(\frac{1}{3}i\pi)$$

Converting to rectangular or polar coordinates simplifies mixed forms.

```
A=1+i
B=sqrt(2)*exp(i*pi/4)
A-B
```

$$1 + i - 2^{1/2}\exp(\frac{1}{4}i\pi)$$

```
rect
```

$$0$$

Rectangular complex quantities, when raised to a power, are multiplied out.

```
(a+i*b)^2
```

$$a^2 - b^2 + 2iab$$

When a and b are numerical and the power is negative, the evaluation is done as follows.

$$(a+ib)^{-n} = \left[\frac{a-ib}{(a+ib)(a-ib)}\right]^n = \left[\frac{a-ib}{a^2+b^2}\right]^n$$

This removes i from the denominator. Here are a few examples.

```
1/(2-i)
```

$$\frac{2}{5} + \frac{1}{5}i$$

```
(-1+3i)/(2-i)
```

$$-1 + i$$

The absolute value of a complex number returns its magnitude.

```
abs(3+4*i)
```

$$5$$

In light of this, the following result might be unexpected.

```
abs(a+b*i)
```

$$\text{abs}(a + ib)$$

The result is not $\sqrt{a^2 + b^2}$ because that would assume that a and b are real. For example, suppose that $a = 0$ and $b = i$. Then

$$|a + ib| = | - 1| = 1$$

and

$$\sqrt{a^2 + b^2} = \sqrt{-1} = i$$

Hence

$$|a + ib| \neq \sqrt{a^2 + b^2} \quad \text{for some } a, b \in \mathcal{C}$$

The *mag* function is an alternative. It treats symbols like a and b as real.

```
mag(a+b*i)
```

$$(a^2 + b^2)^{1/2}$$

$d(f,x)$ returns the derivative of f with respect to x. The x can be omitted for expressions in x.

```
d(x^2)
```

$$2x$$

The following table summarizes the various ways to obtain multiderivatives.

$\dfrac{\partial^2 f}{\partial x^2}$	`d(f,x,x)`	`d(f,x,2)`
$\dfrac{\partial^2 f}{\partial x\,\partial y}$	`d(f,x,y)`	
$\dfrac{\partial^{m+n+\cdots} f}{\partial x^m\,\partial y^n\cdots}$	`d(f,x,...,y,...)`	`d(f,x,m,y,n,...)`

The gradient of f is obtained by using a vector for x in $d(f,x)$.

```
r=sqrt(x^2+y^2)
d(r,(x,y))
```

$$\begin{pmatrix} \dfrac{x}{(x^2+y^2)^{1/2}} \\ \dfrac{y}{(x^2+y^2)^{1/2}} \end{pmatrix}$$

The f in $d(f,x)$ can be a tensor function. Gradient raises the rank by one.

```
F=(x+2y,3x+4y)
X=(x,y)
d(F,X)
```

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$$

The function f in $d(f)$ does not have to be defined. It can be a template function with just a name and an argument list. Eigenmath checks the argument list to figure out what to do. For example, $d(f(x),x)$ evaluates to itself because f depends on x. However, $d(f(x),y)$ evaluates to zero because f does not depend on y.

```
d(f(x),x)
```

$$\partial(f(x),x)$$

```
d(f(x),y)
```

$$0$$

```
d(f(x,y),y)
```

$$\partial(f(x,y),y)$$

```
d(f(),t)
```

$$\partial(f(),t)$$

As the final example shows, an empty argument list causes $d(f)$ to always evaluate to itself, regardless of the second argument.

Template functions are useful for experimenting with differential forms. For example, let us check the identity

$$\text{div}(\text{curl}\,\mathbf{F}) = 0$$

for an arbitrary vector function $\mathbf{F}$.

```
F=(F1(x,y,z),F2(x,y,z),F3(x,y,z))
curl(U)=(d(U[3],y)-d(U[2],z),d(U[1],z)-d(U[3],x),d(U[2],x)-d(U[1],y))
div(U)=d(U[1],x)+d(U[2],y)+d(U[3],z)
div(curl(F))
```

$$0$$

integral(f, x) returns the integral of f with respect to x. The x can be omitted for expressions in x. A multi-integral can be obtained by extending the argument list.

```
integral(x^2)
```

$$\frac{1}{3}x^3$$

```
integral(x*y,x,y)
```

$$\frac{1}{4}x^2y^2$$

defint$(f, x, a, b, \ldots)$ computes the definite integral of f with respect to x evaluated from a to b. The argument list can be extended for multiple integrals.

The following example computes the integral of $f = x^2$ over the domain of a semicircle. For each x along the abscissa, y ranges from 0 to $\sqrt{1-x^2}$.

```
defint(x^2,y,0,sqrt(1-x^2),x,-1,1)
```

$$\frac{1}{8}\pi$$

As an alternative, the *eval* function can be used to compute a definite integral step by step.

```
I=integral(x^2,y)
I=eval(I,y,sqrt(1-x^2))-eval(I,y,0)
I=integral(I,x)
eval(I,x,1)-eval(I,x,-1)
```

$$\frac{1}{8}\pi$$

Here is a useful trick. Difficult integrals involving sine and cosine can often be solved by using exponentials. Trigonometric simplifications involving powers and multiple angles turn into simple algebra in the exponential domain. For example, the definite integral

$$\int_0^{2\pi} \left(\sin^4 t - 2\cos^3(t/2)\sin t\right) dt$$

can be solved as follows.

```
f=sin(t)^4-2*cos(t/2)^3*sin(t)
f=circexp(f)
defint(f,t,0,2*pi)
```

$$-\frac{16}{5}+\frac{3}{4}\pi$$

Here is a check.

```
g=integral(f,t)
f-d(g,t)
```

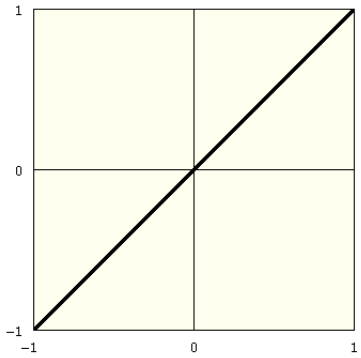
$$0$$

The fundamental theorem of calculus was established by James Gregory, a contemporary of Newton. The theorem is a formal expression of the inverse relation between integrals and derivatives.

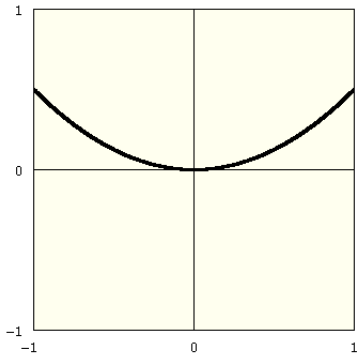
$$\int_a^b f'(x)\,dx = f(b) - f(a)$$

Here is an Eigenmath demonstration of the fundamental theorem of calculus.

```
f=x^2/2
xrange=(-1,1)
yrange=xrange
draw(d(f))
```




```
draw(integral(d(f)))
```



The first graph shows that $f'(x)$ is antisymmetric, therefore the total area under the curve from -1 to 1 sums to zero. The second graph shows that $f(1) = f(-1)$. Hence for $f(x) = \frac{1}{2}x^2$ we have

$$\int_{-1}^1 f'(x)\,dx = f(1) - f(-1) = 0$$

Line integrals are easily computed by converting the coordinates x, y and z into functions of t. This has the effect of changing the measure as well. For instance, dx becomes $(dx/dt)\,dt$. The following line integral problems are from *Advanced Calculus, Fifth Edition* by Wilfred Kaplan.

Evaluate $\int y^2\,dx$ along the straight line from $(0,0)$ to $(2,2)$.

```
x=2t
y=2t
defint(y^2*d(x,t),t,0,1)
```

$$\frac{8}{3}$$

Evaluate $\int y\,dx$ along the straight line from $(2,1)$ to $(1,2)$.

```
x=2-t
y=t+1
defint(y*d(x),t,0,1)
```

$$-\frac{3}{2}$$

Evaluate $\int x\,dy$ along the straight line from $(1,1)$ to $(2,1)$.

```
x=t+1
y=1
defint(x*d(y),t,0,1)
```

$$0$$

Evaluate $\int z\,dx + x\,dy + y\,dz$ along the path $x = 2t+1$, $y = t^2$, $z = 1+t^3$, $0 \le t \le 1$.

```
x=2t+1
y=t^2
z=1+t^3
f=z*d(x)+x*d(y)+y*d(z)
defint(f,t,0,1)
```

$$\frac{163}{30}$$

Let S be a surface parameterized by x and y. That is, let $S = (x, y, z)$ where $z = f(x, y)$. The tangent lines at a point on S form a tiny parallelogram. The area a of the parallelogram is given by the magnitude of the cross product.

$$a = \left| \frac{\partial S}{\partial x} \times \frac{\partial S}{\partial y} \right|$$

By summing over all the parallelograms we obtain the total surface area A. Hence

$$A = \iint dA = \iint a \, dx \, dy$$

The following example computes the surface area of a unit disk parallel to the xy plane.

```
z=2
S=(x,y,z)
a=abs(cross(d(S,x),d(S,y)))
defint(a,y,-sqrt(1-x^2),sqrt(1-x^2),x,-1,1)
```

$$\pi$$

The result is π, the area of a unit circle, which is what we expect. The following example computes the surface area of $z = x^2 + 2y$ over a unit square.

```
z=x^2+2y
S=(x,y,z)
a=abs(cross(d(S,x),d(S,y)))
defint(a,x,0,1,y,0,1)
```

$$\frac{3}{2} + \frac{5}{8} \log(5)$$

As a practical matter, $f(x, y)$ must be very simple in order for Eigenmath to solve the double integral.

Find the area of the spiral ramp defined by[1]

$$S = \begin{pmatrix} u\cos v \\ u\sin v \\ v \end{pmatrix}, \qquad 0 \le u \le 1, \qquad 0 \le v \le 3\pi$$

In this example, the coordinates x, y and z are all functions of an independent parameter space.

```
x=u*cos(v)
y=u*sin(v)
z=v
S=(x,y,z)
a=abs(cross(d(S,u),d(S,v)))
defint(a,u,0,1,v,0,3pi)
```

$$\frac{3}{2}\pi\log(1+2^{1/2}) + \frac{3\pi}{2^{1/2}}$$

```
float
```

10.8177

[1] Williamson and Trotter, *Multivariable Mathematics,* p. 598.

A surface integral is like adding up all the wind on a sail. In other words, we want to compute

$$\iint \mathbf{F} \cdot \mathbf{n} \, dA$$

where $\mathbf{F} \cdot \mathbf{n}$ is the amount of wind normal to a tiny parallelogram dA. The integral sums over the entire area of the sail. Let S be the surface of the sail parameterized by x and y. (In this model, the z direction points downwind.) By the properties of the cross product we have the following for the unit normal $\mathbf{n}$ and for dA.

$$\mathbf{n} = \frac{\frac{\partial S}{\partial x} \times \frac{\partial S}{\partial y}}{\left| \frac{\partial S}{\partial x} \times \frac{\partial S}{\partial y} \right|} \qquad dA = \left| \frac{\partial S}{\partial x} \times \frac{\partial S}{\partial y} \right| dx \, dy$$

Hence

$$\iint \mathbf{F} \cdot \mathbf{n} \, dA = \iint \mathbf{F} \cdot \left(\frac{\partial S}{\partial x} \times \frac{\partial S}{\partial y} \right) dx \, dy$$

Evaluate the surface integral

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma$$

where $\mathbf{F} = xy^2z\mathbf{i} - 2x^3\mathbf{j} + yz^2\mathbf{k}$, S is the surface $z = 1 - x^2 - y^2$, $x^2 + y^2 \leq 1$ and $\mathbf{n}$ is upper.[2]

Note that the surface intersects the xy plane in a circle. By the right hand rule, crossing x into y yields $\mathbf{n}$ pointing upwards hence

$$\mathbf{n}\, d\sigma = \left(\frac{\partial S}{\partial x} \times \frac{\partial S}{\partial y}\right) dx\, dy$$

The following Eigenmath code computes the surface integral. The symbols f and h are used as temporary variables.

```
z=1-x^2-y^2
F=(x*y^2*z,-2*x^3,y*z^2)
S=(x,y,z)
f=dot(F,cross(d(S,x),d(S,y)))
h=sqrt(1-x^2)
defint(f,y,-h,h,x,-1,1)
```

$$\frac{1}{48}\pi$$

[2]Kaplan, *Advanced Calculus,* p. 313.

Green's theorem tells us that

$$\oint P\,dx + Q\,dy = \iint \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

Evaluate $\oint (2x^3 - y^3)\,dx + (x^3 + y^3)\,dy$ around the circle $x^2 + y^2 = 1$ using Green's theorem.[3]

It turns out that Eigenmath cannot solve the double integral over x and y directly. Polar coordinates are used instead.

```
P=2x^3-y^3
Q=x^3+y^3
f=d(Q,x)-d(P,y)
x=r*cos(theta)
y=r*sin(theta)
f=eval(f)
defint(f*r,r,0,1,theta,0,2pi)
```

$$\frac{3}{2}\pi$$

A few words of explanation are in order. The line $f = eval(f)$ is necessary to update f with the polar substitutions for x and y. The $defint$ integrand is $f{*}r$ because $r\,dr\,d\theta = dx\,dy$.

Now let us try computing the line integral side of Green's theorem and see if we get the same result. We need to use the trick of converting sine and cosine to exponentials so that Eigenmath can find a solution.

```
x=cos(t)
y=sin(t)
P=2x^3-y^3
Q=x^3+y^3
f=P*d(x,t)+Q*d(y,t)
f=circexp(f)
defint(f,t,0,2pi)
```

$$\frac{3}{2}\pi$$

[3] Wilfred Kaplan, *Advanced Calculus, 5th Edition,* 287.

Stokes' theorem identifies a special equivalence of line and surface integrals.

$$\oint P\,dx + Q\,dy + R\,dz = \iint_S (\operatorname{curl}\mathbf{F}) \cdot \mathbf{n}\,d\sigma$$

where $\mathbf{F} = (P, Q, R)$. For S parametrized by x and y we have

$$\mathbf{n}\,d\sigma = \left(\frac{\partial S}{\partial x} \times \frac{\partial S}{\partial y}\right) dx\,dy$$

In many cases, converting an integral according to Stokes' theorem can turn a difficult problem into an easy one.

Let $\mathbf{F} = (y, z, x)$ and let S be the part of the paraboloid $z = 4 - x^2 - y^2$ that is above the xy plane. The perimeter of the paraboloid is the circle $x^2 + y^2 = 2$. Calculate both the line and surface integrals. It turns out that we need to use polar coordinates so that *defint* can succeed.

```
--Surface integral
z=4-x^2-y^2
F=(y,z,x)
S=(x,y,z)
f=dot(curl(F),cross(d(S,x),d(S,y)))
x=r*cos(theta)
y=r*sin(theta)
f=eval(f)
defint(f*r,r,0,2,theta,0,2pi)
```

$$-4\pi$$

```
--Line integral
x=2*cos(t)
y=2*sin(t)
z=4-x^2-y^2
P=y
Q=z
R=x
f=P*d(x,t)+Q*d(y,t)+R*d(z,t)
f=circexp(f)
defint(f,t,0,2pi)
```

$$-4\pi$$

The next section of the manual is an eclectic collection of example scripts. The scripts can be copied using the Adobe Reader text selection tool and then pasted into the Eigenmath script window. The examples will be followed by a list of Eigenmath's built-in functions.

François Viète was the first to discover an exact formula for π. Here is his formula.

$$\frac{2}{\pi} = \frac{\sqrt{2}}{2} \times \frac{\sqrt{2+\sqrt{2}}}{2} \times \frac{\sqrt{2+\sqrt{2+\sqrt{2}}}}{2} \times \cdots$$

Let $a_0 = 0$ and $a_n = \sqrt{2 + a_{n-1}}$. Then we can write

$$\frac{2}{\pi} = \frac{a_1}{2} \times \frac{a_2}{2} \times \frac{a_3}{2} \times \cdots$$

Solving for π we have

$$\pi = 2 \times \frac{2}{a_1} \times \frac{2}{a_2} \times \frac{2}{a_3} \times \cdots = 2 \prod_{k=1}^{\infty} \frac{2}{a_k}$$

Let us now use Eigenmath to compute π according to Viète's formula. Of course, we cannot calculate all the way out to infinity, we have to stop somewhere. It turns out that nine factors are just enough to get six digits of accuracy.

```
a(n)=test(n=0,0,sqrt(2+a(n-1)))
float(2*product(k,1,9,2/a(k)))
```

3.14159

The function $a(n)$ calls itself n times so overall there are 54 calls to $a(n)$. By using a different algorithm with temporary variables, we can get the answer in just nine steps.

```
a=0
b=2
for(k,1,9,a=sqrt(2+a),b=b*2/a)
float(b)
```

3.14159

The curl of a vector function can be expressed in tensor form as

$$\operatorname{curl} \mathbf{F} = \epsilon_{ijk} \frac{\partial F_k}{\partial x_j}$$

where ϵ_{ijk} is the Levi-Civita tensor. The following script demonstrates that this formula is equivalent to computing curl the old fashioned way. First, define ϵ_{ijk}.

```
epsilon=zero(3,3,3)
epsilon[1,2,3]=1
epsilon[2,3,1]=1
epsilon[3,1,2]=1
epsilon[3,2,1]=-1
epsilon[1,3,2]=-1
epsilon[2,1,3]=-1
```

Next, define a generic vector function $\mathbf{F}$ and then compute $A = \epsilon_{ijk}\, \partial F_k/\partial x_j$. The first summation is over k which corresponds to indices 3 and 4. The second summation is over j which (with k out of the way) corresponds to indices 2 and 3.

```
F=(FX(),FY(),FZ())
A=outer(epsilon,d(F,(x,y,z)))
A=contract(A,3,4)
A=contract(A,2,3)
```

Now compute curl the old fashioned way and check for equality.

```
B=(
 d(F[3],y)-d(F[2],z),
 d(F[1],z)-d(F[3],x),
 d(F[2],x)-d(F[1],y)
)
```

```
A-B
```

$$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

The following is a variation on the previous script. The product $\epsilon_{ijk}\,\partial F_k/\partial x_j$ is computed in just one line of code. In addition, there is an optimization. The outer product and the first contraction have been replaced with a dot product.

```
F=(FX(),FY(),FZ())

epsilon=zero(3,3,3)
epsilon[1,2,3]=1
epsilon[2,3,1]=1
epsilon[3,1,2]=1
epsilon[3,2,1]=-1
epsilon[1,3,2]=-1
epsilon[2,1,3]=-1

A=contract(dot(epsilon,d(F,(x,y,z))),2,3)

B=(
 d(F[3],y)-d(F[2],z),
 d(F[1],z)-d(F[3],x),
 d(F[2],x)-d(F[1],y)
)

--Are A and B equal? Subtract to find out.

A-B
```

For total energy E, kinetic energy K and potential energy V we have

$$E = K + V$$

The corresponding formula for a quantum harmonic oscillator is

$$(2n + 1)\psi = -\frac{d^2\psi}{dx^2} + x^2\psi$$

where n is an integer and represents the quantization of energy values. The solution to the above equation is

$$\psi_n(x) = \exp(-x^2/2)H_n(x)$$

where $H_n(x)$ is the nth Hermite polynomial in x. The following Eigenmath code checks $E = K + V$ for $n = 7$.

```
n=7
psi=exp(-x^2/2)*hermite(x,n)
E=(2*n+1)*psi
K=-d(psi,x,x)
V=x^2*psi
E-K-V
```

$$0$$

Hydrogen wavefunctions ψ are solutions to the differential equation

$$\frac{\psi}{n^2} = \nabla^2\psi + \frac{2\psi}{r}$$

where n is an integer representing the quantization of total energy and r is the radial distance of the electron. The Laplacian operator in spherical coordinates is

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}$$

The general form of ψ is

$$\psi = r^l e^{-r/n} L_{n-l-1}^{2l+1}(2r/n) P_l^{|m|}(\cos\theta) e^{im\phi}$$

where L is a Laguerre polynomial, P is a Legendre polynomial and l and m are integers such that

$$1 \le l \le n-1, \qquad -l \le m \le l$$

The general form can be expressed as the product of a radial wavefunction R and a spherical harmonic Y.

$$\psi = RY, \qquad R = r^l e^{-r/n} L_{n-l-1}^{2l+1}(2r/n), \qquad Y = P_l^{|m|}(\cos\theta) e^{im\phi}$$

The following code checks $E = K + V$ for $n, l, m = 7, 3, 1$.

```
laplacian(f)=1/r^2*d(r^2*d(f,r),r)+
1/(r^2*sin(theta))*d(sin(theta)*d(f,theta),theta)+
1/(r*sin(theta))^2*d(f,phi,phi)
n=7
l=3
m=1
R=r^l*exp(-r/n)*laguerre(2*r/n,n-l-1,2*l+1)
Y=legendre(cos(theta),l,abs(m))*exp(i*m*phi)
psi=R*Y
E=psi/n^2
K=laplacian(psi)
V=2*psi/r
simplify(E-K-V)
```

0

The space shuttle accelerates from zero to 17,000 miles per hour in 8 minutes. A Corvette accelerates from zero to 60 miles per hour in 4.5 seconds. The following script compares the two.

```
vs=17000*"mile"/"hr"
ts=8*"min"/(60*"min"/"hr")
as=vs/ts
as
vc=60*"mile"/"hr"
tc=4.5*"sec"/(3600*"sec"/"hr")
ac=vc/tc
ac
"Time for Corvette to reach orbital velocity:"
vs/ac
vs/ac*60*"min"/"hr"
```

Here is the result when the script runs. It turns out that the space shuttle accelerates more than twice as fast as a Corvette.

$$a_s = \frac{127500 \text{ mile}}{(\text{hr})^2}$$

$$a_c = \frac{48000 \text{ mile}}{(\text{hr})^2}$$

```
Time for Corvette to reach orbital velocity:
```

0.354167 hr

21.25 min

abs

abs(x) returns the absolute value or vector length of x. The mag function should be used for complex x.

```
P=(x,y)
abs(P)
```

$$(x^2 + y^2)^{1/2}$$

adj

adj(m) returns the adjunct of matrix m.

and

and($a, b, \ldots$) returns the logical "and" of predicate expressions.

arccos

arccos(x) returns the inverse cosine of x.

arccosh

arccosh(x) returns the inverse hyperbolic cosine of x.

arcsin

arcsin(x) returns the inverse sine of x.

arcsinh

arcsinh(x) returns the inverse hyperbolic sine of x.

arctan

arcttan(x) returns the inverse tangent of x.

arctanh

arctanh(x) returns the inverse hyperbolic tangent of x.

arg

arg(z) returns the angle of complex z.

ceiling

ceiling(x) returns the smallest integer not less than x.

check

check(x) In a script, if the predicate x is true then continue, else stop.

choose

choose(n, k) returns $\binom{n}{k}$

circexp

circexp(x) returns expression x with circular functions converted to exponential forms. Sometimes this will simplify an expression.

coeff

coeff(p, x, n) returns the coefficient of x^n in polynomial p.

cofactor

cofactor(m, i, j) returns of the cofactor of matrix m with respect to row i and column j.

conj

conj(z) returns the complex conjugate of z.

contract

contract(a, i, j) returns tensor a summed over indices i and j.

cos

cos(x) returns the cosine of x.

cosh

cosh(x) returns the hyperbolic cosine of x.

d

d(f, x) returns the derivative of f with respect to x.

defint

defint$(f, x, a, b, \ldots)$ returns the definite integral of f with respect to x evaluated from a to b. The argument list can be extended for multiple integrals. For example, $d(f, x, a, b, y, c, d)$.

deg

deg(p, x) returns the degree of polynomial p in x.

denominator

denominator(x) returns the denominator of expression x.

det

det(m) returns the determinant of matrix m.

display

display(x) evaluates expression x and displays the result. Useful for printing from inside a "for" loop.

do

do$(a, b, \ldots)$ evaluates the argument list from left to right. Returns the result of the last argument.

dot

dot$(a, b, \ldots)$ returns the dot product of tensors.

draw

draw(f, x) draws the function f with respect to x.

erf

erf(x) returns the error function of x.

erfc

erf(x) returns the complementary error function of x.

eval

eval(f, x, n) returns f evaluated at $x = n$.

exp

exp(x) returns e^x.

expcos

expcos(x) returns the cosine of x in exponential form.

```
expcos(x)
```

$$\frac{1}{2}\exp(-ix) + \frac{1}{2}\exp(ix)$$

expsin

expsin(x) returns the sine of x in exponential form.

```
expsin(x)
```

$$\frac{1}{2}i\exp(-ix) - \frac{1}{2}i\exp(ix)$$

factor

factor(n) factors the integer n.

```
factor(12345)
```

$$3 \times 5 \times 823$$

factor(p, x) factors polynomial p in x. The last argument can be omitted for polynomials in x.

```
factor(125*x^3-1)
```

$$(5x-1)(25x^2+5x+1)$$

factorial

Example:

```
10!
```

$$3628800$$

filter

filter($f, a, b, \ldots$) returns f with terms involving a, b, etc. removed.

```
1/a+1/b+1/c
```

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}$$

```
filter(last,a)
```

$$\frac{1}{b}+\frac{1}{c}$$

float

float(x) converts x to a floating point value.

```
sum(n,0,20,(-1/2)^n)
```

$$\frac{699051}{1048576}$$

```
float(last)
```

$$0.666667$$

floor

floor(x) returns the largest integer not greater than x.

for

for($i, j, k, a, b, \ldots$) For i equals j through k evaluate a, b, etc.

```
x=0
y=2
for(k,1,9,x=sqrt(2+x),y=2*y/x)
float(y)
```

$$3.14159$$

gcd

gcd($a, b, \ldots$) returns the greatest common divisor.

hermite

hermite(x, n) returns the nth Hermite polynomial in x.

hilbert

hilbert(n) returns a Hilbert matrix of order n.

imag

imag(z) returns the imaginary part of complex z.

inner

inner($a, b, \ldots$) returns the inner product of tensors. Same as the dot product.

integral

integral(f, x) returns the integral of f with respect to x.

inv

inv(m) returns the inverse of matrix m.

isprime

isprime(n) returns 1 if n is prime, zero otherwise.

```
isprime(2^53-111)
```


laguerre

laguerre(x, n, a) returns the nth Laguerre polynomial in x. If a is omitted then $a = 0$ is used.

lcm

lcm($a, b, \ldots$) returns the least common multiple.

legendre

legendre(x, n, m) returns the nth Legendre polynomial in x. If m is omitted then $m = 0$ is used.

log

log(x) returns the natural logarithm of x.

mag

mag(z) returns the magnitude of complex z.

mod

mod(a, b) returns the remainder of a divided by b.

not

not(x) negates the result of predicate expression x.

numerator

numerator(x) returns the numerator of expression x.

or

or($a, b, \ldots$) returns the logical "or" of predicate expressions.

outer

outer($a, b, \ldots$) returns the outer product of tensors.

polar

polar(z) converts complex z to polar form.

prime

prime(n) returns the nth prime number, $1 \leq n \leq 10{,}000$.

print

print(x) evaluates expression x and displays the result in typewriter style. Useful for printing from inside a "for" loop.

product

product(i, j, k, f) returns $\displaystyle\prod_{i=j}^{k} f$

quote

quote(x) returns expression x unevaluated.

quotient

quotient(p, q, x) returns the quotient of polynomials in x.

rank

rank(a) returns the number of indices that tensor a has. A scalar has no indices so its rank is zero.

rationalize

rationalize(x) puts everything over a common denominator.

```
rationalize(a/b+b/a)
```

$$\frac{a^2+b^2}{ab}$$

real

real(z) returns the real part of complex z.

rect

rect(z) returns complex z in rectangular form.

roots

roots(p, x) returns the values of x such that the polynomial $p(x) = 0$. The polynomial should be factorable over integers.

simplify

simplify(x) returns x in a simpler form.

sin

sin(x) returns the sine of x.

sinh

sinh(x) returns the hyperbolic sine of x.

sqrt

sqrt(x) returns the square root of x.

stop

In a script, it does what it says.

subst

subst(a, b, c) substitutes a for b in c and returns the result.

sum

sum(i, j, k, f) returns $\sum_{i=j}^{k} f$

tan

tan(x) returns the tangent of x.

tanh

tanh(x) returns the hyperbolic tangent of x.

taylor

taylor(f, x, n, a) returns the Taylor expansion of f of x at a. The argument n is the degree of the expansion. If a is omitted then $a = 0$ is used.

```
taylor(1/cos(x),x,4)
```

$$\frac{5}{24}x^4 + \frac{1}{2}x^2 + 1$$

test

test($a, b, c, d, \ldots$) If a is true then b is returned else if c is true then d is returned, etc. If the number of arguments is odd then the last argument is returned when all else fails.

trace

trace(m) returns the trace of matrix m.

transpose

transpose(a, i, j) returns the transpose of tensor a with respect to indices i and j. If i and j are omitted then 1 and 2 are used. Hence a matrix can be transposed with a single argument.

```
A=((a,b),(c,d))
transpose(A)
```

$$\begin{pmatrix} a & c \\ b & d \end{pmatrix}$$

unit

unit(n) returns an $n \times n$ identity matrix.

```
unit(2)
```

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

zero

zero$(i, j, \ldots)$ returns a null tensor with dimensions i, j, etc. Useful for creating a tensor and then setting the component values.

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