Set Theory

CSE 215, Foundations of Computer Science

Stony Brook University

http://www.cs.stonybrook.edu/~cse215

Set theory

- Abstract set theory is one of the foundations of mathematical thought
  - Most mathematical objects (e.g. numbers) can be defined in terms of sets
- Let S denote a set:
  - a ∈ S means that a is an element of S
    - Example: $1 \in \{1,2,3\}$, $3 \in \{1,2,3\}$
  - a ∉ S means that a is not an element of S
    - Example: $4 \notin \{1,2,3\}$
  - If S is a set and P(x) is a property that elements of S may or may not satisfy: $A = \{x \in S \mid P(x)\}$ is the set of all elements x of S such that P(x)

Subsets: Proof and Disproof

- Def.: $A \subseteq B \Leftrightarrow \forall x$, if $x \in A$ then $x \in B$

  (it is a formal universal conditional statement)
- Negation: $A \nsubseteq B \Leftrightarrow \exists x$ such that $x \in A$ and $x \notin B$
- A is a **proper subset of** B ($A \subset B$) $\Leftrightarrow$

  (1) $A \subseteq B$ AND

  (2) there is at least one element in B that is not in A
- Examples:

  $\{1\} \subseteq \{1\}$

  $\{1\} \subset \{1, 2\}$

  $\{1\} \subseteq \{1, \{1\}\}$

  $\{1\} \subset \{1, \{1\}\}$

Set Theory

- **Element Argument:** The Basic Method for Proving That One Set Is a **Subset** of Another

Let sets X and Y be given. To prove that $\mathbf{X \subseteq Y}$,

1. Suppose that x is a particular **[but arbitrarily chosen]** element of X,
2. show that x is also an element of Y.

Set Theory

- Example of an Element Argument Proof: **$A \subseteq B$?**

  $A = \{m \in Z \mid m = 6r + 12 \text{ for some } r \in Z\}$

  $B = \{n \in Z \mid n = 3s \text{ for some } s \in Z\}$

  Suppose x is a particular but arbitrarily chosen element of A.

  [We must show that $x \in B$].

  By definition of A, there is an integer r such that

  $x = 6r + 12 \Leftrightarrow x = 3(2r + 4)$

  But, $s = 2r + 4$ is an integer because products and sums of integers are integers.

  $x=3s$. ➔ By definition of B, x is an element of B.

  $A \subseteq B$

Set Theory

- Disprove $\mathbf{B} \subseteq \mathbf{A}$: $\mathbf{B} \nsubseteq \mathbf{A}$.

  $A = \{m \in \mathbf{Z} \,|\, m = 6r + 12 \text{ for some } r \in \mathbf{Z}\}$

  $B = \{n \in \mathbf{Z} \mid n = 3s \text{ for some } s \in \mathbf{Z}\}$

  Disprove = show that the statement $\mathbf{B} \subseteq \mathbf{A}$ is false.

  We must find an element of B (x=3s) that is not an element of A (x=6r+12).

  Let $x = 3 = 3 * 1 \rightarrow 3 \in B$

  $3 \in A$? We assume by contradiction $\exists r \in \mathbf{Z}$, such that: $6r+12=3$ (assumption) $\rightarrow 2r + 4 = 1 \rightarrow 2r = -3 \rightarrow r=-3/2$

  But $r=-3/2$ is not an integer($\notin \mathbf{Z}$). Thus, contradiction $\rightarrow$ $3 \notin A$.

  $3 \in B$ and $3 \notin A$, so $\mathbf{B} \nsubseteq \mathbf{A}$.

Set Equality

- A = B, if, and only if, every element of A is in B and every element of B is in A.

  $A = B \iff A \subseteq B$ and $B \subseteq A$

- Example:

  $A = \{m \in Z \mid m = 2a \text{ for some integer } a\}$

  $B = \{n \in Z \mid n = 2b - 2 \text{ for some integer } b\}$

  - Proof Part 1: $A \subseteq B$

  Suppose x is a particular but arbitrarily chosen element of A.

  By definition of A, there is an integer a such that $x = 2a$

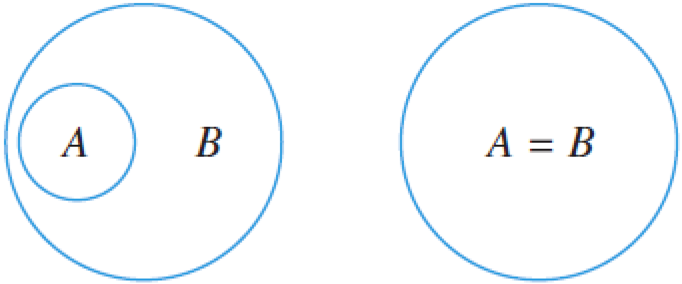
  Let $b = a + 1$, $2b - 2 = 2(a + 1) - 2 = 2a + 2 - 2 = 2a = x$

  Thus, $x \in B$.

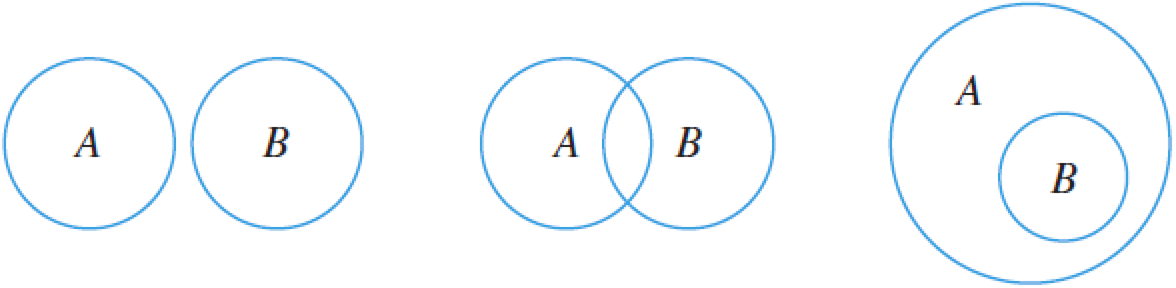
  - Proof Part 2: $B \subseteq A$ (proved in similar manner)

Venn Diagrams

- $A \subseteq B$

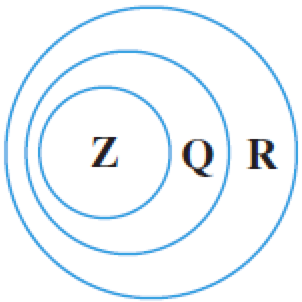


- $A \not\subseteq B$



Relations among Sets of Numbers

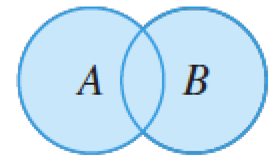
- **Z**, **Q**, and **R** denote the sets of integers, rational numbers, and real numbers
- **Z** ⊆ **Q** because every integer is rational (any integer n can be written in the form n/1)
  - **Z** is a proper subset of **Q**: there are rationals that are not integers (e.g., 1/2)
- **Q** ⊆ **R** because every rational is real
  - **Q** is a proper subset of **R** because there are real numbers that are not rational (e.g., $\sqrt{2}$)



Operations on Sets

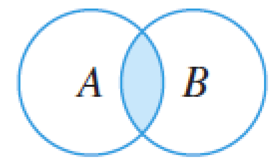
Let A and B be subsets of a universal set U.

1. The union of A and B: $A \cup B$ is the set of all elements that are in at least one of A or B:



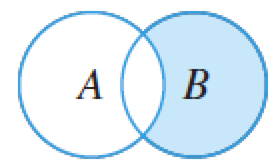
$$A \cup B = \{x \in U \mid x \in A \text{ or } x \in B\}$$

2. The intersection of A and B: $A \cap B$ is the set of all elements that are common to both A and B.



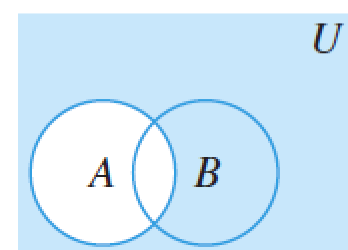
$$A \cap B = \{x \in U \mid x \in A \text{ and } x \in B\}$$

3. The difference of B minus A (relative complement of A in B): $B-A$ (or $B\backslash A$) is the set of all elements that are in B and not A.



$$B - A = \{x \in U \mid x \in B \text{ and } x \notin A\}$$

4. The complement of A: A^c is the set of all elements in U that are not in A.



$$A^c = \{x \in U \mid x \notin A\}$$

Operations on Sets

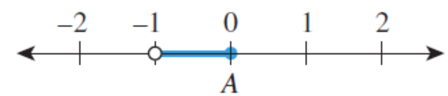
- Example: Let U = {a, b, c, d, e, f, g} and let A = {a, c, e, g} and B = {d, e, f, g}.
  - $A \cup B = \{a, c, d, e, f, g\}$
  - $A \cap B = \{e, g\}$
  - $B - A = \{d, f\}$
  - $A^c = \{b, d, f\}$

Subsets of real numbers

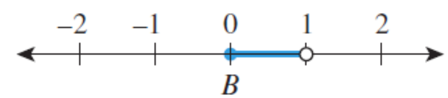
- Given real numbers a and b with $a \leq b$:
  - $(a, b) = \{x \in R \mid a < x < b\}$
  - $(a, b] = \{x \in R \mid a < x \leq b\}$
  - $[a, b) = \{x \in R \mid a \leq x < b\}$
  - $[a, b] = \{x \in R \mid a \leq x \leq b\}$
- The symbols ∞ and $-\infty$ are used to indicate intervals that are unbounded either on the right or on the left:
  - $(a,\infty)=\{x \in R \mid a < x\}$
  - $[a,\infty) =\{x \in R \mid a \leq x\}$
  - $(-\infty, b)=\{x \in R \mid x < b\}$
  - $(-\infty, b]=\{x \in R \mid x \leq b\}$

Subsets of real numbers

- Example: Let



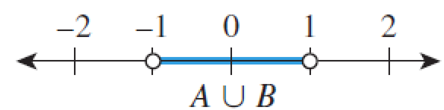
$A = (-1, 0] = \{x \in R \mid -1 < x \leq 0\}$



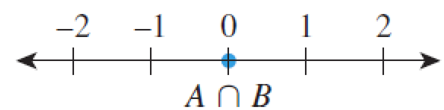
$B = [0, 1) = \{x \in R \mid 0 \leq x < 1\}$

$A \cup B = \{x \in R \mid x \in (-1, 0] \text{ or } x \in [0, 1)\}$

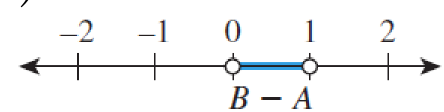
$= \{x \in R \mid x \in (-1, 1)\} = (-1, 1)$



$A \cap B = \{x \in R \mid x \in (-1, 0] \text{ and } x \in [0, 1)\} = \{0\}.$

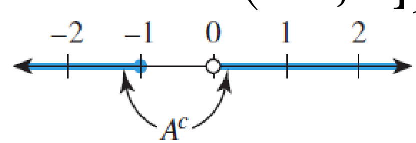


$B - A = \{x \in R \mid x \in [0, 1) \text{ and } x \notin (-1, 0]\} = (0, 1)$



$A^c = \{x \in R \mid \text{it is not the case that } x \in (-1, 0]\}$

$= (-\infty, -1] \cup (0, \infty)$



Set theory

- **Unions and Intersections of an Indexed Collection of Sets**
  - Given sets $A_0, A_1, A_2, \ldots$ that are subsets of a universal set U and given a nonnegative integer n (set sequence)
  - $\bigcup_{i=0}^{n} A_i = \{x \in U \mid x \in A_i \text{ for at least one } i = 0, 1, 2, \ldots, n\}$
  - $\bigcup_{i=1}^{\infty} A_i = \{x \in U \mid x \in A_i \text{ for at least one nonnegative integer } i\}$
  - $\bigcap_{i=0}^{n} A_i = \{x \in U \mid x \in A_i \text{ for all } i = 0, 1, 2, \ldots, n\}$
  - $\bigcap_{i=1}^{\infty} A_i = \{x \in U \mid x \in A_i \text{ for all nonnegative integers } i\}$

Indexed Sets

- Example: for each positive integer i,

$A_i = \{x \in \mathbf{R} \mid -1/i < x < 1/i\} = (-1/i\ ,\ 1/i)$

  - $A_1 \cup A_2 \cup A_3 = \{x \in \mathbf{R} \mid$ x is in at least one of the intervals $(-1,1), (-1/2, 1/2), (-1/3, 1/3)\ \} = (-1, 1)$
  - $A_1 \cap A_2 \cap A_3 = \{x \in \mathbf{R} \mid$ x is in all of the intervals $(-1,1), (-1/2,1/2), (-1/3, 1/3)\ \} = (-1/3, 1/3)$
  - $\bigcup_{i=1}^{\infty} A_i = \{x \in \mathbf{R} \mid$ x is in at least one of the intervals $(-1/i,1/i)$ where i is a positive integer$\} = (-1, 1)$
  - $\bigcap_{i=1}^{\infty} A_i = \{x \in \mathbf{R} \mid$ x is in all of the intervals $(-1/i,1/i)$, where i is a positive integer$\} = \{0\}$

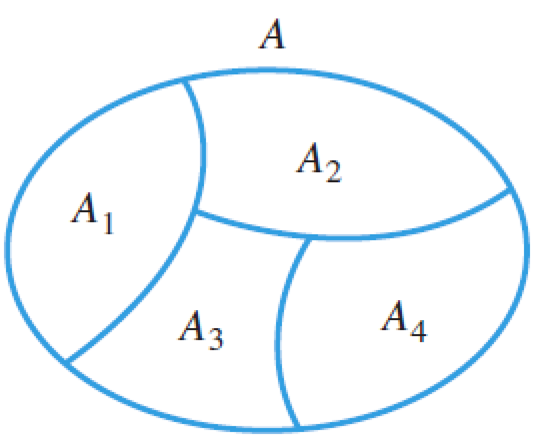
The Empty Set ∅ ({})

- ∅ = {} a set that has no elements
- Examples:
  - $\{1,2\} \cap \{3,4\} = \emptyset$
  - $\{x \in R \mid 3 < x < 2\} = \emptyset$

Partitions of Sets

- A and B are ***disjoint*** $\Leftrightarrow A \cap B = \emptyset$
  - the sets A and B have no elements in common
- Sets $A_1, A_2, A_3, \ldots$ are ***mutually disjoint*** (pairwise disjoint or non-overlapping) $\Leftrightarrow$ no two sets A_i and A_j $(i \neq j)$ have any elements in common
  - $\forall\, i,j = 1,2,3,\ldots,\ i \neq j \rightarrow A_i \cap A_j = \emptyset$
- A finite or infinite collection of nonempty sets $\{A_1, A_2, A_3, \ldots\}$ is a ***partition*** of a set A $\Leftrightarrow$

  1. $A = \bigcup_{i=1}^{\infty} A_i$
  2. $A_1, A_2, A_3, \ldots$ are mutually disjoint



Partitions of Sets

- Examples:
  - $A = \{1, 2, 3, 4, 5, 6\}$

$A_1 = \{1, 2\}$ $A_2 = \{3, 4\}$ $A_3 = \{5, 6\}$

$\{A_1, A_2, A_3\}$ is a partition of A:

- $A = A_1 \cup A_2 \cup A_3$

- A_1, A_2 and A_3 are mutually disjoint:

$$A_1 \cap A_2 = A_1 \cap A_3 = A_2 \cap A_3 = \emptyset$$

  - $T_1 = \{n \in \mathbf{Z} \mid n = 3k, \text{ for some integer } k\}$

$T_2 = \{n \in \mathbf{Z} \mid n = 3k + 1, \text{ for some integer } k\}$

$T_3 = \{n \in \mathbf{Z} \mid n = 3k + 2, \text{ for some integer } k\}$

$\{T_1, T_2, T_3\}$ is a partition of $\mathbf{Z}$

Power Set

- Given a set A, the ***power set*** of A, P(A), is the set of all subsets of A
  - Examples:

$$P(\{x, y\}) = \{\emptyset, \{x\}, \{y\}, \{x, y\}\}$$

$$P(\emptyset) = \{\emptyset\}$$

$$P(\{\emptyset\}) = \{\emptyset, \{\emptyset\}\}$$

Cartesian Product

- An **ordered n-tuple** $(x_1, x_2, \ldots, x_n)$ consists of the elements $x_1, x_2, \ldots, x_n$ together with the ordering: first x_1, then x_2, and so forth up to x_n
- Two ordered n-tuples $(x_1, x_2, \ldots, x_n)$ and $(y_1, y_2, \ldots, y_n)$ are ***equal:***

  $(x_1, x_2, \ldots, x_n) = (y_1, y_2, \ldots, y_n) \Leftrightarrow x_1 = y_1$ and $x_2 = y_2$ and … $x_n = y_n$
- The **Cartesian product** of $A_1, A_2, \ldots, A_n$:

  $A_1 \times A_2 \times \ldots \times A_n = \{(a_1, a_2, \ldots, a_n) \mid a_1 \in A_1, a_2 \in A_2, \ldots, a_n \in A_n\}$
- Example: $A=\{1,2\}$, $B=\{3,4\}$

  $A \times B = \{(1,3), (1,4), (2,3), (2,4)\}$

Cartesian Product

- Example: let $A = \{x, y\}$, $B = \{1, 2, 3\}$, and $C = \{a, b\}$

$A \times B \times C = \{(u,v,w) \mid u \in A, v \in B, \text{ and } w \in C\}$

$= \{(x, 1, a), (x, 2, a), (x, 3, a), (y, 1, a), (y, 2, a), (y, 3, a), (x, 1, b), (x, 2, b), (x, 3, b), (y, 1, b), (y, 2, b), (y, 3, b)\}$

$(A \times B) \times C = \{(u,v) \mid u \in A \times B \text{ and } v \in C\}$

$= \{((x, 1), a), ((x, 2), a), ((x, 3), a), ((y, 1), a), ((y, 2), a), ((y, 3), a), ((x, 1), b), ((x, 2), b), ((x, 3), b), ((y, 1), b), ((y, 2), b), ((y, 3), b)\}$

Supplemental: Algorithm to Check Subset

Input: m, n [positive integers], a,b [one-dimensional arrays]

Algorithm Body:

```
i := 1,    answer := "A ⊆ B"
while (i ≤ m   and   answer = "A ⊆ B")
    j := 1,   found := "no"
    while (j ≤ n   and   found = "no")
        if a[i] = b[j] then found := "yes"
        j := j + 1
    end while
    if found = "no" then answer := "A ⊈ B"
    i := i + 1
end while
```

Output: answer [a string]: "A ⊆ B" or "A ⊈ B"

Properties of Sets

- Inclusion of Intersection:

  $A \cap B \subseteq A$ and $A \cap B \subseteq B$

- Inclusion in Union:

  $A \subseteq A \cup B$ and $B \subseteq A \cup B$

- Transitive Property of Subsets:

  $A \subseteq B$ and $B \subseteq C \rightarrow A \subseteq C$

- $x \in A \cup B \Leftrightarrow x \in A$ or $x \in B$
- $x \in A \cap B \Leftrightarrow x \in A$ and $x \in B$
- $x \in B - A \Leftrightarrow x \in B$ and $x \notin A$
- $x \in A^c \Leftrightarrow x \notin A$
- $(x, y) \in A \times B \Leftrightarrow x \in A$ and $y \in B$

Proof of a Subset Relation

- For all sets A and B, **$A \cap B \subseteq A$**.

The statement to be proved is universal:

$$\forall \text{ sets } A \text{ and } B,\ A \cap B \subseteq A$$

Suppose A and B are any (particular but arbitrarily chosen) sets.

$A \cap B \subseteq A$, we must show $\forall x,\ x \in A \cap B \rightarrow x \in A$

Suppose x is any (particular but arbitrarily chosen) element in $A \cap B$.

By definition of $A \cap B$, $x \in A$ and $x \in B$.

Therefore, $\therefore x \in A$ **Q.E.D.**

Set Identities

- For all sets A, B, and C:
  - Commutative Laws: $A \cup B = B \cup A$ and $A \cap B = B \cap A$
  - Associative Laws: $(A \cup B) \cup C = A \cup (B \cup C)$ and $(A \cap B) \cap C = A \cap (B \cap C)$
  - Distributive Laws: $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$, $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
  - Identity Laws: $A \cup \emptyset = A$ and $A \cap U = A$
  - Complement Laws: $A \cup A^c = U$ and $A \cap A^c = \emptyset$
  - Double Complement Law: $(A^c)^c = A$
  - Idempotent Laws: $A \cup A = A$ and $A \cap A = A$
  - Universal Bound Laws: $A \cup U = U$ and $A \cap \emptyset = \emptyset$
  - De Morgan's Laws: $(A \cup B)^c = A^c \cap B^c$ and $(A \cap B)^c = A^c \cup B^c$
  - Absorption Laws: $A \cup (A \cap B) = A$ and $A \cap (A \cup B) = A$
  - Complements of U and $\emptyset$: $U^c = \emptyset$ and $\emptyset^c = U$
  - Set Difference Law: $A - B = A \cap B^c$

Proof of a Set Identity

- For all sets A, B, and C, $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

Suppose A, B, and C are arbitrarily chosen sets.

1. $A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C)$

Show: $\forall x$, if $x \in A \cup (B \cap C)$ then $x \in (A \cup B) \cap (A \cup C)$

Suppose $x \in A \cup (B \cap C)$, arbitrarily chosen. (1)

We must show $x \in (A \cup B) \cap (A \cup C)$.

From (1), by definition of union, $x \in A$ or $x \in B \cap C$

Case 1.1: $x \in A$. By definition of union: $x \in A \cup B$ and $x \in A \cup C$

By definition of intersection: $x \in (A \cup B) \cap (A \cup C)$. (2)

Case 1.2: $x \in B \cap C$. By definition of intersection: $x \in B$ and $x \in C$

By definition of union: $x \in A \cup B$ and $x \in A \cup C$. And (2) again.

2. $(A \cup B) \cap (A \cup C) \subseteq A \cup (B \cap C)$ (proved in similar manner)

Proof of a De Morgan's Law for Sets

- For all sets A and B: $(A \cup B)^c = A^c \cap B^c$

Suppose A and B are arbitrarily chosen sets.

(➔) Suppose $x \in (A \cup B)^c$.

By definition of complement: $x \notin A \cup B$

it is false that (x is in A or x is in B)

By De Morgan's laws of logic: x is **not** in A **and** x is **not** in B.

$x \notin A$ and $x \notin B$

Hence $x \in A^c$ and $x \in B^c$

$x \in A^c \cap B^c$

(🡐) Proved in similar manner.

Intersection and Union with a Subset

- For any sets A and B, if $A \subseteq B$, then $A \cap B = A$ and $A \cup B = B$

$A \cap B = A \Leftrightarrow$ (1) $A \cap B \subseteq A$ and (2) $A \subseteq A \cap B$

(1) $A \cap B \subseteq A$ is true by the inclusion of intersection property

(2) Suppose $x \in A$ (arbitrary chosen).

From $A \subseteq B$, then $x \in B$ (by definition of subset relation).

From $x \in A$ and $x \in B$, thus $x \in A \cap B$ (by definition of $\cap$)

$A \subseteq A \cap B$

$A \cup B = B \Leftrightarrow$ (3) $A \cup B \subseteq B$ and (4) $B \subseteq A \cup B$

(3) and (4) proved in similar manner to (1) and (2)

The Empty Set

- **A Set with No Elements Is a Subset of Every Set:**

  If E is a set with no elements and A is any set, then $E \subseteq A$

Proof (by contradiction): Suppose there exists an empty set E with no elements and a set A such that $E \nsubseteq A$.

By definition of $\nsubseteq$: there is an element of E ($x \in E$) that is not an element of A ($x \notin A$).

Contradiction with E was empty, so $x \notin E$. **Q.E.D.**

- **Uniqueness of the Empty Set:** There is only one set with no elements.

Proof: Suppose E_1 and E_2 are both sets with no elements.

By the above property: $E_1 \subseteq E_2$ and $E_2 \subseteq E_1$ ➔ $E_1 = E_2$ **Q.E.D.**

The Element Method

- To prove that a set $X = \emptyset$, prove that X has no elements by contradiction:
  - suppose X has an element and derive a contradiction.
- Example 1: For any set A, $A \cap \emptyset = \emptyset$.

Proof: Let A be a particular (arbitrarily chosen) set.

$A \cap \emptyset = \emptyset \Leftrightarrow A \cap \emptyset$ has no elements

Proof by contradiction: suppose there is x such that $x \in A \cap \emptyset$.

By definition of intersection, $x \in A$ and $x \in \emptyset$

Contradiction since $\emptyset$ has no elements **Q.E.D.**

The Element Method

- Example 2: For all sets A, B, and C,

  if $A \subseteq B$ and $B \subseteq C^c$, then $A \cap C = \emptyset$.

Proof: Suppose A, B, and C are any sets such that

$A \subseteq B$ and $B \subseteq C^c$

Suppose there is an element $x \in A \cap C$.

By definition of intersection, $x \in A$ and $x \in C$.

From $x \in A$ and $A \subseteq B$, by definition of subset, $x \in B$.

From $x \in B$ and $B \subseteq C^c$, by definition of subset, $x \in C^c$.

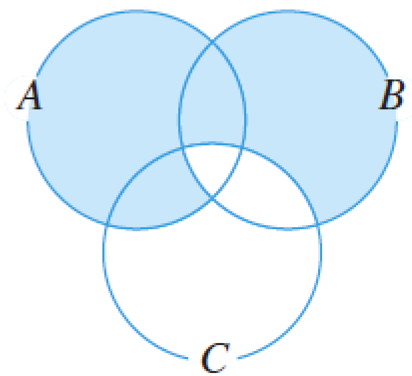
By definition of complement $x \notin C$ (contradiction with $x \in C$).

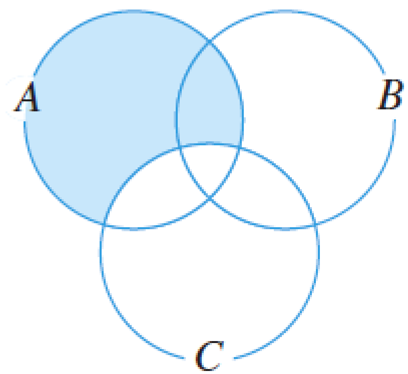
Q.E.D.

Disproofs

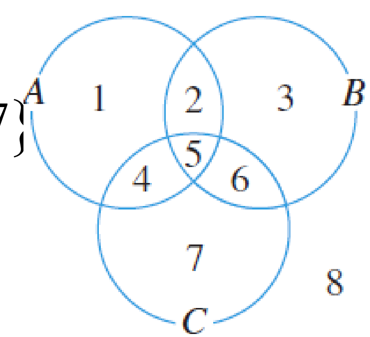
- Disproving an alleged set property amounts to finding a counterexample for which the property is false.
- Example: Disprove that for all sets A,B, and C,

$(A-B)\cup(B-C) \quad = \quad A-C \ ?$





The property is false ⇔ there are sets A, B, and C for which the equality does not hold



Counterexample 1: $A=\{1,2,4,5\}, B=\{2,3,5,6\}, C=\{4,5,6,7\}$

$(A-B)\cup(B-C)=\{1,4\}\cup\{2,3\}=\{1,2,3,4\} \neq \{1,2\}=A-C$

Counterexample 2: $A=\emptyset, B=\{1\}, C=\emptyset$

Cardinality of a set

- The cardinality of a set A: N(A) or $|A|$ is a measure of the "number of elements of the set"
- Example: $|\{2, 4, 6\}| = 3$
- For any sets A and B,

$$|A \cup B| + |A \cap B| = |A| + |B|$$

- If A and B are disjoint sets, then

$$|A \cup B| = |A| + |B|$$

The Size of the Power Set

- For all int. $n \geq 0$, X has n elements → P(X) has 2^n elements.

Proof (by mathematical induction): Q(n): Any set with n elements has 2^n subsets.

Q(0): Any set with 0 elements has 2^0 subsets:

The power set of the empty set Ø is the set P(Ø) = {Ø}.

P(Ø) has $1=2^0$ element: the empty set Ø.

For all integers $k \geq 0$, if Q(k) is true then Q(k+1) is also true.

Q(k): Any set with k elements has 2^k subsets.

We show Q(k+1): Any set with k +1elements has 2^{k+1} subsets.

Let X be a set with k+1 elements and z∈X (since X has at least one element).

X−{z} has k elements, so P(X−{z}) has 2^k elements.

Any subset A of X−{z} is a subset of X: A ∈ P(X).

Any subset A of X−{z}, can also be matched with {z}: A∪{z} ∈ P(X)

All subsets A and A∪{z} are all the subsets of X ➔ P(X) has $2*2^k=2^{k+1}$ elements

Algebraic Proofs of Set Identities

- **Algebraic Proofs = Use of laws to prove new identities**
  1. **Commutative Laws: $A \cup B = B \cup A$ and $A \cap B = B \cap A$**
  2. **Associative Laws: $(A \cup B) \cup C = A \cup (B \cup C)$ and $(A \cap B) \cap C = A \cap (B \cap C)$**
  3. **Distributive Laws: $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ and $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$**
  4. **Identity Laws: $A \cup \emptyset = A$ and $A \cap U = A$**
  5. **Complement Laws: $A \cup A^c = U$ and $A \cap A^c = \emptyset$**
  6. **Double Complement Law: $(A^c)^c = A$**
  7. **Idempotent Laws: $A \cup A = A$ and $A \cap A = A$**
  8. **Universal Bound Laws: $A \cup U = U$ and $A \cap \emptyset = \emptyset$**
  9. **De Morgan's Laws: $(A \cup B)^c = A^c \cap B^c$ and $(A \cap B)^c = A^c \cup B^c$**
  10. **Absorption Laws: $A \cup (A \cap B) = A$ and $A \cap (A \cup B) = A$**
  11. **Complements of U and $\emptyset$: $U^c = \emptyset$ and $\emptyset^c = U$**
  12. **Set Difference Law: $A - B = A \cap B^c$**

Algebraic Proofs of Set Identities

- Example: for all sets A, B, and C, $(A \cup B) - C = (A - C) \cup (B - C)$.

Algebraic proof:

$(A \cup B) - C = (A \cup B) \cap C^c$ by the set difference law

$= C^c \cap (A \cup B)$ by the commutative law for $\cap$

$= (C^c \cap A) \cup (C^c \cap B)$ by the distributive law

$= (A \cap C^c) \cup (B \cap C^c)$ by the commutative law for $\cap$

$= (A - C) \cup (B - C)$ by the set difference law.

Algebraic Proofs of Set Identities

- Example: for all sets A and B, $A - (A \cap B) = A - B$.

$A - (A \cap B) = A \cap (A \cap B)^c$ by the set difference law

$= A \cap (A^c \cup B^c)$ by De Morgan's laws

$= (A \cap A^c) \cup (A \cap B^c)$ by the distributive law

$= \emptyset \cup (A \cap B^c)$ by the complement law

$= (A \cap B^c) \cup \emptyset$ by the commutative law for $\cup$

$= A \cap B^c$ by the identity law for $\cup$

$= A - B$ by the set difference law.

Correspondence between logical equivalences and set identities

Logical Equivalences	Set Properties
For all statement variables p, q, and r:	For all sets A, B, and C:
a. $p \vee q \equiv q \vee p$ b. $p \wedge q \equiv q \wedge p$	a. $A \cup B = B \cup A$ b. $A \cap B = B \cap A$
a. $p \wedge (q \wedge r) \equiv p \wedge (q \wedge r)$ b. $p \vee (q \vee r) \equiv p \vee (q \vee r)$	a. $A \cup (B \cup C) = A \cup (B \cup C)$ b. $A \cap (B \cap C) = A \cap (B \cap C)$
a. $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$ b. $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$	a. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ b. $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
a. $p \vee \mathbf{c} \equiv p$ b. $p \wedge \mathbf{t} \equiv p$	a. $A \cup \emptyset = A$ b. $A \cap U = A$
a. $p \vee \sim p \equiv \mathbf{t}$ b. $p \wedge \sim p \equiv \mathbf{c}$	a. $A \cup A^c = U$ b. $A \cap A^c = \emptyset$
$\sim(\sim p) \equiv p$	$(A^c)^c = A$
a. $p \vee p \equiv p$ b. $p \wedge p \equiv p$	a. $A \cup A = A$ b. $A \cap A = A$
a. $p \vee \mathbf{t} \equiv \mathbf{t}$ b. $p \wedge \mathbf{c} \equiv \mathbf{c}$	a. $A \cup U = U$ b. $A \cap \emptyset = \emptyset$
a. $\sim(p \vee q) \equiv \sim p \wedge \sim q$ b. $\sim(p \wedge q) \equiv \sim p \vee \sim q$	a. $(A \cup B)^c = A^c \cap B^c$ b. $(A \cap B)^c = A^c \cup B^c$
a. $p \vee (p \wedge q) \equiv p$ b. $p \wedge (p \vee q) \equiv p$	a. $A \cup (A \cap B) = A$ b. $A \cap (A \cup B) = A$
a. $\sim \mathbf{t} \equiv \mathbf{c}$ b. $\sim \mathbf{c} \equiv \mathbf{t}$	a. $U^c = \emptyset$ b. $\emptyset^c = U$

Boolean Algebra

- ∨ (or) corresponds to ∪ (union)
- ∧ (and) corresponds to ∩ (intersection)
- ~ (negation) corresponds to ^{c} (complementation)
- t (a tautology) corresponds to U (a universal set)
- c (a contradiction) corresponds to ∅ (the empty set)

- Logic and sets are special cases of the same general structure Boolean algebra.

Boolean Algebra

- A Boolean algebra is a set B together with two operations + and ·, such that for all a and b in B both a + b and a ·b are in B and the following properties hold:

1. Commutative Laws: For all a and b in B, $a+b=b+a$ and $a\cdot b=b\cdot a$
2. Associative Laws: For all a,b, and c in B,
$$(a+b)+c=a+(b+c) \text{ and } (a\cdot b)\cdot c=a\cdot(b\cdot c)$$
3. Distributive Laws: For all a, b, and c in B, $a+(b\cdot c)=(a+b)\cdot(a+c)$ and $a\cdot(b+c)=(a\cdot b)+(a\cdot c)$
4. Identity Laws: There exist distinct elements 0 and 1 in B such that for all a in B, $a+0=a$ and $a\cdot 1=a$
5. Complement Laws: For each a in B, there exists an element in B, $\overline{a}$, complement or negation of a, such that $a+\overline{a}=1$ and $a\cdot\overline{a}=0$

Properties of a Boolean Algebra

- Uniqueness of the Complement Law: For all a and x in B, if $a+x=1$ and $a \cdot x=0$ then $x=\overline{a}$
- Uniqueness of 0 and 1: If there exists x in B such that $a+x=a$ for all a in B, then $x=0$, and if there exists y in B such that $a \cdot y=a$ for all a in B, then $y=1$.
- Double Complement Law: For all $a \in B$, $\overline{(\overline{a})} = a$
- Idempotent Law: For all $a \in B$, $a+a=a$ and $a \cdot a=a$.
- Universal Bound Law: For all $a \in B$, $a+1=1$ and $a \cdot 0 = 0$.
- De Morgan's Laws: For all a and $b \in B$, $\overline{a+b}=\overline{a} \cdot \overline{b}$ and $\overline{a \cdot b}=\overline{a}+\overline{b}$
- Absorption Laws: For all a and $b \in B$, $(a+b) \cdot a=a$ and $(a \cdot b)+a=a$
- Complements of 0 and 1: $\overline{0} = 1$ and $\overline{1} = 0$.

Properties of a Boolean Algebra

- Uniqueness of the Complement Law: For all a and x in B, if $a+x=1$ and $a \cdot x=0$ then $x=\overline{a}$

Proof: Suppose a and x are particular (arbitrarily chosen) in B that satisfy the hypothesis: $a+x=1$ and $a \cdot x=0$.

$x = x \cdot 1$	because 1 is an identity for $\cdot$
$= x \cdot (a + \overline{a})$	by the complement law for +
$= x \cdot a + x \cdot \overline{a}$	by the distributive law for $\cdot$ over +
$= a \cdot x + x \cdot \overline{a}$	by the commutative law for $\cdot$
$= 0 + x \cdot \overline{a}$	by hypothesis
$= a \cdot \overline{a} + x \cdot \overline{a}$	by the complement law for $\cdot$
$= (\overline{a} \cdot a) + (\overline{a} \cdot x)$	by the commutative law for $\cdot$
$= \overline{a} \cdot (a + x)$	by the distributive law for $\cdot$ over +
$= \overline{a} \cdot 1$	by hypothesis
$= \overline{a}$	because 1 is an identity for $\cdot$

Russell's Paradox

- Most sets are not elements of themselves.
- Imagine a set A being an element of itself $A \in A$.
- Let S be the set of all sets that are not elements of themselves:

$$S = \{A \mid A \text{ is a set and } A \notin A\}$$

- **Is S an element of itself? Yes&No contradiction.**
  - If $S \in S$, then S does not satisfy the defining property for S: $S \notin S$.
  - If $S \notin S$, then satisfies the defining property for S, which implies that: $S \in S$.

The Barber Puzzle

- In a town there is a male barber who shaves all those men, and only those men, who do not shave themselves.
- Question: Does the barber shave himself?
  - If the barber shaves himself, he is a member of the class of men who shave themselves. The barber does not shave himself because he doesn't shave men who shave themselves.
  - If the barber does not shave himself, he is a member of the class of men who do not shave themselves. The barber shaves every man in this class, so the barber must shave himself. Both Yes&No derive contradiction!

Russell's Paradox

- One possible solution: except powersets, whenever a set is defined using a predicate as a defining property, the set is a subset of a ***known*** set.
  - Then S (form Russell's Paradox) is not a set in the universe of sets.

The Halting Problem

- There is no computer algorithm that will accept any algorithm X and data set D as input and then will output "halts" or "loops forever" to indicate whether or not X terminates in a finite number of steps when X is run with data set D.

Proof sketch (by contradiction): Suppose there is an algorithm CheckHalt such that for any input algorithm X and a data set D, it prints "halts" or "loops forever".

A new algorithm Test(X)

loops forever if CheckHalt(X, X) prints "halts" or

stops if CheckHalt(X, X) prints "loops forever".

Test(Test) = ?

- If Test(Test) terminates after a finite number of steps, then the value of CheckHalt(Test, Test) is "halts" and so Test(Test) loops forever. Contradiction!
- If Test(Test) does not terminate after a finite number of steps, then CheckHalt(Test, Test) prints "loops forever" and so Test(Test) terminates. Contradiction!

So, CheckHalt doesn't exist.